

CORRECT SOLVABILITY OF THE STURM-LIOUVILLE EQUATION WITH DELAYED ARGUMENT

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ABSTRACT. We consider the equation

$$-y''(x) + q(x)y(x - \varphi(x)) = f(x), \quad x \in \mathbb{R} \quad (1)$$

where $f \in C(\mathbb{R})$ and

$$0 \leq \varphi \in C^{\text{loc}}(R), \quad 1 \leq q \in C^{\text{loc}}(\mathbb{R}). \quad (2)$$

Here $C^{\text{loc}}(\mathbb{R})$ is the set of functions continuous in every point of the number axis. By a solution of (1), we mean any function y , doubly continuously differentiable everywhere in \mathbb{R} , which satisfies (1). We show that under certain additional conditions on the functions φ and q to (2), (1) has a unique solution y , satisfying the inequality

$$\|y\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}$$

where the constant $c \in (0, \infty)$ does not depend on the choice of $f \in C(\mathbb{R})$.

1. INTRODUCTION

In the present paper, we consider the equation

$$-y''(x) + q(x)y(x - \varphi(x)) = f(x), \quad x \in \mathbb{R} \quad (1.1)$$

where $f \in C(\mathbb{R})$ and

$$0 \leq \varphi \in C^{\text{loc}}(R), \quad 1 \leq q \in C^{\text{loc}}(\mathbb{R}). \quad (1.2)$$

By the symbol $C^{\text{loc}}(\mathbb{R})$, we denote the set of functions continuous in every point of the number axis \mathbb{R} .

By a solution of (1.1) we mean any doubly continuously differentiable function $y(x)$, satisfying (1.1) for all $x \in \mathbb{R}$. In addition, we say that equation (1.1) is correctly solvable in $C(\mathbb{R})$ if the following assertions hold:

- I) for every function $f \in C(\mathbb{R})$ equation (1.1) has a unique solution $y \in C(\mathbb{R})$;
- II) there is a constant $c \in (0, \infty)$ such that regardless of the choice of $f \in C(\mathbb{R})$, the solution $y \in C(\mathbb{R})$ of (1.1) satisfies the inequality

$$\|y\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}. \quad (1.3)$$

Our goal is to study, in a conceptual way, the problem of correct solvability of equation (1.1) in the space $C(\mathbb{R})$. (For brevity, below we say “the question on I)–II”, or “the problem

I)–II)”). Such an investigation is needed because this is the first time the problem I)–II) is being posed. Indeed, to the best of our knowledge, for equations with delayed argument there has been studied initial and boundary value problems on a finite segment or on a semi-axis (see [1, 7, 11, 12, 13]). However, the special feature of problem I)–II) is that equation (1.1) is considered on the whole axis, and requirements to its solutions are imposed apart from I)–II). Therefore, the main result of the paper is statement asserting that problem I)–II) makes sense, i.e., the set of equations (1.1) correctly solvable in $C(\mathbb{R})$ is non-empty. This statement follows from the following theorem which is our main result.

Theorem 1.1. *If, together with (1.2), the following two conditions hold:*

1) *there is a constant $a \geq 1$ such that for all $x \in \mathbb{R}$ the following inequalities hold:*

$$a^{-1}q(x) \leq q(t) \leq aq(x) \quad \text{for } \forall t \in [x-1, x+1]; \quad (1.4)$$

2)

$$\sigma \leq 1/6\sqrt{a}, \quad \text{where } \sigma \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} (\varphi(x)q(x)), \quad (1.5)$$

then equation (1.1) is correctly solvable in $C(\mathbb{R})$. In addition, equation (1.1) is separable in $C(\mathbb{R})$, i.e., there is a constant $c \in (0, \infty)$ such that regardless of the choice of $f \in C(\mathbb{R})$, the solution $y \in C(\mathbb{R})$ of (1.1) satisfies the inequality

$$\|y''(x)\|_{C(\mathbb{R})} + \|q(x)y(x - \varphi(x))\|_{C(\mathbb{R})} \leq c\|f(x)\|_{C(\mathbb{R})}. \quad (1.6)$$

Corollary 1.2. *There is a constant $c \in (0, \infty)$ such that the solution $y \in C(\mathbb{R})$ of (1.1) satisfies the estimate*

$$\|q(x)y(x)\|_{C(\mathbb{R})} \leq c\|f(x)\|_{C(\mathbb{R})}, \quad \forall f \in C(\mathbb{R}). \quad (1.7)$$

The paper is constructed as follows. In §2, we collect the information needed for the proofs; in §3, we present some auxiliary assertions; §4 contains a proof of Theorem 1.1; and, finally, in §5 we give an example of this theorem.

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2. PRELIMINARIES

The information presented below is used in the proofs. Here and throughout the sequel, we assume that conditions (1.2) are satisfied. They are not referred to and do not appear in the statements.

Theorem 2.1. [4, 6] *There exists a fundamental system of solutions (FSS) $\{u(x), v(x)\}$, $x \in \mathbb{R}$ of the equation*

$$z''(x) = q(x)z(x), \quad x \in \mathbb{R} \quad (2.1)$$

which has the following properties:

$$\begin{aligned} u(x) > 0, \quad v(x) > 0, \quad u'(x) < 0, \quad v'(x) > 0, \quad x \in \mathbb{R}, \\ v'(x)u(x) - u'(x)v(x) = 1, \quad x \in \mathbb{R} \end{aligned} \quad (2.2)$$

$$\begin{aligned} \lim_{x \rightarrow -\infty} v(x) = \lim_{x \rightarrow -\infty} v'(x) = \lim_{x \rightarrow \infty} u(x) = \lim_{x \rightarrow \infty} u'(x) = 0 \\ \lim_{x \rightarrow \infty} v(x) = \lim_{x \rightarrow \infty} v'(x) = \lim_{x \rightarrow -\infty} u(x) = \lim_{x \rightarrow -\infty} |u'(x)| = \infty. \end{aligned}$$

$$|p'(x)| < 1, \quad x \in \mathbb{R}; \quad \rho(x) = u(x)v(x), \quad x \in \mathbb{R}, \quad (2.3)$$

$$\frac{|u'(x)|}{u(x)} = \frac{1 - \rho'(x)}{2\rho(x)}, \quad \frac{v'(x)}{v(x)} = \frac{1 + \rho'(x)}{2\rho(x)}, \quad x \in \mathbb{R}. \quad (2.4)$$

Lemma 2.2. [5] *For a given $x \in \mathbb{R}$ consider the following equations in $d \geq 0$:*

$$\int_0^{\sqrt{2}d} \int_{x-t}^x q(\xi) d\xi dt = 1, \quad \int_0^{\sqrt{2}d} \int_x^{x+t} q(\xi) d\xi dt = 1. \quad (2.5)$$

Each of equations (2.5) has a unique finite positive solution.

Denote by $d_1(x)$, $d_2(x)$, $x \in \mathbb{R}$, the solutions of equations (2.5), respectively.

Theorem 2.3. [5] *We have the inequalities*

$$\frac{1}{\sqrt{2}} \leq \frac{|u'(x)|}{u(x)} d_2(x); \quad \frac{v'(x)}{v(x)} d_1(x) \leq \sqrt{2}, \quad x \in \mathbb{R}, \quad (2.6)$$

$$\frac{1}{\sqrt{2}} \frac{d_1(x)d_2(x)}{d_1(x) + d_2(x)} \leq \rho(x) \leq \sqrt{2} \frac{d_1(x)d_2(x)}{d_1(x) + d_2(x)}, \quad x \in \mathbb{R}, \quad (2.7)$$

Consider the equation

$$-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R}. \quad (2.8)$$

By a solution of (2.8) we mean any doubly continuously differentiable function $y(x)$ satisfying (2.8) for all $x \in \mathbb{R}$.

Definition 2.4. [2] *We say that equation (2.8) is correctly solvable in $C(\mathbb{R})$ if the following conditions are satisfied:*

- a) *for every function $f \in C(\mathbb{R})$ there is a unique solution $y \in C(\mathbb{R})$ of equation (2.8);*
- b) *there is a constant $c \in (0, \infty)$ such that regardless of the choice of $f \in C(\mathbb{R})$, the solution $y \in C(\mathbb{R})$ of (2.8) satisfies the estimate*

$$\|y\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}. \quad (2.9)$$

Remark 2.5. By c , $c(\cdot)$, we denote absolute positive constants which are not essential for exposition and may differ even with a single chain of calculations.

Theorem 2.6. [2] *Equation (2.8) is correctly solvable in $C(\mathbb{R})$. Its solution $y \in C(\mathbb{R})$ is of the form*

$$y(x) = (Gf)(x) = \int_{-\infty}^{\infty} G(x, t)f(t)dt, \quad x \in \mathbb{R}. \quad (2.10)$$

Here $G(x, t)$ is the Green function of equation (2.8):

$$G(x, t) = \begin{cases} u(x)v(t), & x \geq t \\ u(t)v(x), & x \leq t \end{cases} \quad (2.11)$$

Denote

$$\mathcal{D}(\mathbb{R}) = \{y \in C(\mathbb{R}) : y \in C_{\text{loc}}^{(2)}(\mathbb{R}), -y''(x) + q(x)y(x) \in C(\mathbb{R}), x \in \mathbb{R}\}, \quad (2.12)$$

$$(\mathcal{L}y)(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R}, \quad y \in \mathcal{D}(\mathbb{R}). \quad (2.13)$$

Here $C_{\text{loc}}^{(2)}(\mathbb{R})$ is the set of functions doubly continuously differentiable for $x \in \mathbb{R}$.

Theorem 2.7. [2] *The operator $\mathcal{L} : \mathcal{D}(\mathbb{R}) \rightarrow C(\mathbb{R})$ is continuously invertible. We have the equality (see (2.10)):*

$$\mathcal{L}^{-1} = G. \quad (2.14)$$

Definition 2.8. [3] *We say that equation (2.8) is separable in $C(\mathbb{R})$ if there is a constant $c \in (0, \infty)$ such that regardless of the choice of $f \in C(\mathbb{R})$, the solution $y \in C(\mathbb{R})$ of (2.8) satisfies the inequality*

$$\|y''\|_{C(\mathbb{R})} + \|qy\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}. \quad (2.15)$$

Remark 2.9. The problem of separating the operator \mathcal{L} into summands:

$$\|\mathcal{L}y\|_{C(\mathbb{R})} \leq \|y''\|_{C(\mathbb{R})} + \|qy\|_{C(\mathbb{R})} \leq c\|\mathcal{L}y\|_{C(\mathbb{R})}, \quad \forall y \in \mathcal{D}(\mathbb{R})$$

was first studied in [8, 9] in the space $L_2(\mathbb{R})$.

Theorem 2.10. [10, pp. 84–85] *Let $\varphi(x)$, $x \in \mathbb{R}$ be a non-negative, continuous function, and let $y(x)$, $x \in \mathbb{R}$ be a doubly continuously differentiable function. Then we have the equality*

$$y(x) = y(x - \varphi(x)) + \varphi(x)y'(x) - \int_{x-\varphi(x)}^x y''(\xi)(\xi - x + \varphi(x))d\xi, \quad x \in \mathbb{R}. \quad (2.16)$$

3. AUXILIARY ASSERTIONS

Some of the statements presented below are interesting in their own right.

Lemma 3.1. *We have the inequalities (see (2.5))*

$$\sup_{x \in \mathbb{R}} d_1(x) \leq 1, \quad \sup_{x \in \mathbb{R}} d_2(x) \leq 1. \quad (3.1)$$

Proof. From (1.2) and (2.5) it follows that

$$1 = \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \geq \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x 1 d\xi \geq d_1^2(x), \quad x \in \mathbb{R} \Rightarrow (3.1).$$

The second inequality in (3.1) can be checked similarly. \square

Lemma 3.2. *We have the inequalities*

$$\|\mathcal{L}^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq 1, \quad (3.2)$$

$$\left\| \frac{d}{dx} \mathcal{L}^{-1} \right\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq \sqrt{2}. \quad (3.3)$$

Proof. In the following relations we only use Theorem 2.1:

$$\left. \begin{aligned} v'(x) &= v'(x) - v'(-\infty) = \int_{-\infty}^x v''(t) dt = \int_{-\infty}^x q(t) v(t) dt \\ -u'(x) &= u'(\infty) - u'(x) = \int_x^{\infty} u''(t) dt = \int_x^{\infty} q(t) u(t) dt \end{aligned} \right\} \Rightarrow$$

$$1 = v'(x)u(x) - u'(x)v(x) = u(x) \int_{-\infty}^x q(t) v(t) dt + v(x) \int_x^{\infty} q(t) u(t) dt$$

$$= \int_{-\infty}^{\infty} q(t) G(x, t) dt \geq \int_{-\infty}^{\infty} G(x, t) dt.$$

This implies that (see (2.14))

$$\|\mathcal{L}^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} = \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} G(x, t) dt \leq 1 \Rightarrow (3.2).$$

In the following relations, together with Theorem 2.1, we use (2.6), (2.14) and (3.1):

$$\begin{aligned} \left\| \frac{d}{dx} \mathcal{L}^{-1} \right\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} \left| \frac{d}{dx} \int_{-\infty}^{\infty} G(x, t) dt \right| = \sup_{x \in \mathbb{R}} \left| u'(x) \int_{-\infty}^x v(t) dt + v'(x) \int_x^{\infty} u(t) dt \right| \\ &\leq \sup_{x \in \mathbb{R}} \left[|u'(x)| \int_{-\infty}^x \frac{v(t)}{v'(t)} \cdot v'(t) dt + v'(x) \int_x^{\infty} \frac{|u(t)|}{|u'(t)|} \cdot |u'(t)| dt \right] \\ &\leq \sqrt{2} \sup_{x \in \mathbb{R}} \left[|u'(x)| \int_{-\infty}^x d_1(t) v'(t) dt + v'(x) \int_x^{\infty} d_2(t) (-u'(t)) dt \right] \\ &\leq \sqrt{2} \sup_{x \in \mathbb{R}} \left[|u'(x)| \int_{-\infty}^x v'(t) dt - v'(x) \int_x^{\infty} u'(t) dt \right] \\ &= \sqrt{2} \sup_{x \in \mathbb{R}} [v'(x)u(x) - u'(x)v(x)] = \sqrt{2}. \end{aligned}$$

□

Lemma 3.3. *Under condition (1.4), we have the inequalities*

$$\frac{1}{\sqrt{2a}} \frac{1}{\sqrt{q(x)}} \leq d_1(x), d_2(x) \leq \sqrt{2a} \frac{1}{\sqrt{q(x)}}, \quad x \in \mathbb{R}. \quad (3.4)$$

Proof. Estimates (3.4) for $d_1(x)$ and $d_2(x)$ are proved in the same way; therefore, here we only consider $d_1(x)$. For a given $x \in \mathbb{R}$, consider the equation in $d \geq 0$:

$$F(d) = 1, \quad F(d) = d \cdot \int_{x-d}^x q(\xi) d\xi. \quad (3.5)$$

Clearly, on $[0, \infty)$ the function $F(d)$ is monotone increasing, and $F(d) \geq d^2$. Since $F(0) = 0$, $F(\infty) = \infty$, we conclude that (3.5) has a unique positive solution.

Denote this solution by $\hat{d}(x)$. Clearly, $\hat{d}(x) \leq 1$ for $x \in \mathbb{R}$ because

$$1 = \hat{d}(x) \cdot \int_{x-\hat{d}(x)}^x q(\xi) d\xi \geq \hat{d}(x) \int_{x-\hat{d}(x)}^x 1 dt = \hat{d}^2(x).$$

In addition, from the first mean value theorem and (1.4), it follows that

$$1 = \hat{d}(x) \int_{x-\hat{d}(x)}^x q(t) dt = q(\tilde{x}) \hat{d}^2(x), \quad \tilde{x} \in [x-1, x] \Rightarrow$$

$$\hat{d}(x) = \frac{1}{\sqrt{q(\tilde{x})}} = \sqrt{\frac{q(x)}{q(\tilde{x})}} \frac{1}{\sqrt{q(x)}} \leq \sqrt{\frac{a}{q(x)}}, \quad x \in \mathbb{R}, \quad (3.6)$$

$$\hat{d}(x) = \frac{1}{\sqrt{q(\tilde{x})}} = \sqrt{\frac{q(x)}{q(\tilde{x})}} \frac{1}{\sqrt{q(x)}} \geq \frac{1}{\sqrt{aq(x)}}, \quad x \in \mathbb{R}. \quad (3.7)$$

Now, from (2.5), properties of $F(d)$, $d \geq 0$, and (3.6) and (3.7), it follows that

$$1 = \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \leq \sqrt{2}d_1(x) \int_{x-\sqrt{2}d_1(x)}^x q(\xi) d\xi = F(\sqrt{2}d_1(x)) \Rightarrow$$

$$\sqrt{2}d_1(x) \geq \hat{d}(x) \geq \frac{1}{\sqrt{aq(x)}} \Rightarrow d_1(x) \geq \frac{1}{\sqrt{2aq(x)}},$$

$$1 = \int_0^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi \geq \int_{\frac{d_1(x)}{\sqrt{2}}}^{\sqrt{2}d_1(x)} \int_{x-t}^x q(\xi) d\xi dt \geq \frac{d_1(x)}{\sqrt{2}} \int_{x-\frac{d_1(x)}{\sqrt{2}}}^x q(\xi) d\xi$$

$$= F\left(\frac{d_1(x)}{\sqrt{2}}\right) \Rightarrow \frac{d_1(x)}{\sqrt{2}} \leq \hat{d}(x) \leq \frac{a}{\sqrt{q(x)}} \Rightarrow d_1(x) \leq \sqrt{\frac{2a}{q(x)}}, \quad x \in \mathbb{R} \Rightarrow (3.5).$$

□

Corollary 3.4. *For $x \in \mathbb{R}$ we have the inequalities*

$$\frac{v(x)}{v(x-d_1(x))} \leq \exp(2\sqrt{2}a^{3/2}); \quad \frac{u(x)}{u(x+d_2(x))} \leq \exp(2\sqrt{2}a^{3/2}). \quad (3.8)$$

Proof. Below we consecutively use (2.6), (3.4), (3.1), (1.4) and once again (3.4):

$$\begin{aligned}
\ln \frac{v(x)}{v(x-d_1(x))} &= \int_{x-d_1(x)}^x \frac{v'(\xi)}{v(\xi)} d\xi \leq \sqrt{2} \int_{x-d_1(x)}^x \frac{d\xi}{d_1(\xi)} \\
&\leq 2\sqrt{a} \int_{x-d_1(x)}^x \sqrt{q(\xi)} d\xi = 2\sqrt{a} \int_{x-d_1(x)}^x \sqrt{\frac{q(\xi)}{q(x)}} \cdot \sqrt{q(x)} d\xi \\
&\leq 2a\sqrt{q(x)}d_1(x) \leq 2\sqrt{2}a^{3/2} \Rightarrow (3.8).
\end{aligned}$$

□

Theorem 3.5. *Let $r \in C^{\text{loc}}(\mathbb{R})$. Then we have the estimates*

$$\frac{1}{2\sqrt{2}a} \exp(-2\sqrt{2}a^{3/2})m_0(r, q) \leq \|r\mathcal{L}^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq 4am_0(r, q), \quad (3.9)$$

$$\left\| r \frac{d}{dx} \mathcal{L}^{-1} \right\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq 8a^{3/2}m_1(r, q). \quad (3.10)$$

Here

$$m_0(r, q) = \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)}; \quad m_1(r, q) = \sup_{x \in \mathbb{R}} \frac{r(x)}{\sqrt{q(x)}}. \quad (3.11)$$

In particular, equation (2.8) is separable in $C(\mathbb{R})$, and we have the inequalities

$$\frac{1}{2\sqrt{2}a} \exp(-2\sqrt{2}a^{3/2}) \leq \|q\mathcal{L}^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq 4a, \quad (3.12)$$

$$\left\| \frac{d^2}{dx^2} \mathcal{L}^{-1} \right\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq 4a + 1. \quad (3.13)$$

Proof. We need the following useful assertion.

Lemma 3.6. *For $x \in \mathbb{R}$ we have the inequality*

$$\int_{-\infty}^{\infty} G(x, t) dt \leq 4 \int_{x-1}^{x+1} G(x; t) dt. \quad (3.14)$$

Proof. Let us check the relations

$$\int_{-\infty}^x v(t) dt \leq 4 \int_{x-1}^x v(t) dt, \quad \int_x^{\infty} u(t) dt \leq 4 \int_x^{x+1} u(t) dt. \quad (3.15)$$

These inequalities are proved in the same way; therefore, below we only consider the second one. Denote

$$z(t) = e^{-t}, \quad t \in \mathbb{R}.$$

The following relations are deduced from Theorem 2.1:

$$\begin{aligned}
& u''(\xi) = q(\xi)u(\xi), \quad z''(\xi) = z(\xi), \quad \xi \in \mathbb{R} \\
\Rightarrow & [u'(\xi)z(\xi) - z'(\xi)u(\xi)]' = u''(\xi)z(\xi) - z''(\xi)u(\xi) = (q(\xi) - 1)u(\xi)z(\xi) \geq 0 \\
\Rightarrow & -[u'(t)z(t) - z'(t)u(t)] = \int_t^\infty [u'(\xi)z(\xi) - z'(\xi)u(\xi)]' d\xi \geq 0, \quad t \in \mathbb{R} \\
\Rightarrow & u'(t)z(t) - z'(t)u(t) \leq 0, \quad t \in \mathbb{R} \quad \Rightarrow \quad u'(t) \leq -u(t), \quad t \in \mathbb{R} \\
\Rightarrow & \frac{u(t)}{u(x)} \leq e^{x-t} \quad \text{as} \quad t \geq x, \quad x \in \mathbb{R}. \tag{3.16}
\end{aligned}$$

Let $x \in \mathbb{R}$, $x_n = x + n$, $n = 1, 2, \dots$. Below we use Theorem 2.1 and (3.16)

$$\begin{aligned}
\int_x^\infty u(t)dt &= \int_x^{x_1} u(t)dt + \sum_{n=1}^\infty \int_{x_n}^{x_{n+1}} u(t)dt \\
&= \int_x^{x_1} u(t)dt \left[1 + \sum_{k=1}^\infty \left(\int_{x_k}^{x_{k+1}} u(t)dt \right) \left(\int_x^{x_1} u(t)dt \right)^{-1} \right] \\
&\leq \int_{x_1}^x u(t)dt \cdot \left[1 + \sum_{n=1}^\infty \frac{u(x_n)}{u(x_1)} \right] \leq \int_x^{x_1} u(t)dt \cdot \left[1 + \sum_{n=1}^\infty e^{-(n-1)} \right] \\
&= \int_x^{x_1} u(t)dt \left[1 + \frac{1}{1 - e^{-1}} \right] \leq 4 \int_x^{x_1} u(t)dt \quad \Rightarrow \quad (3.15).
\end{aligned}$$

Let us now go to (3.12). Below we use Theorem 2.1, (2.5), (2.14), (2.7), (3.8) and (3.4):

$$\begin{aligned}
\|r\mathcal{L}^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} |r(x)| \int_{-\infty}^\infty G(x, t)dt \geq \sup_{x \in \mathbb{R}} |r(x)| \int_{x-d_1(x)}^{x+d_2(x)} G(, t)dt \\
&= \sup_{x \in \mathbb{R}} |r(x)| \left[u(x) \int_{x-d_1(x)}^x v(t)dt + v(x) \int_x^{x+d_2(x)} u(t)dt \right] \\
&\geq \sup_{x \in \mathbb{R}} |r(x)| [u(x)v(x-d_1(x))d_1(x) + v(x)u(x+d_2(x))d_2(x)] \\
&= \sup_{x \in \mathbb{R}} |r(x)|\rho(x) \left[\frac{v(x-d_1(x))}{v(x)}d_1(x) + \frac{u(x+d_2(x))}{u(x)}d_2(x) \right] \\
&\geq \exp(-2\sqrt{2}a^{3/2}) \sup_{x \in \mathbb{R}} |r(x)|\rho(x)(d_1(x) + d_2(x)) \\
&\geq \frac{1}{\sqrt{2}} \exp(-2\sqrt{2}a^{3/2}) \sup_{x \in \mathbb{R}} |r(x)|d_1(x)d_2(x) \\
&\geq \frac{1}{2\sqrt{2}a} \exp(-2\sqrt{2}a^{3/2}) \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} = \frac{\exp(-2\sqrt{2}a^{3/2})}{2\sqrt{2}a} m_0(r, q).
\end{aligned}$$

Below, in the proof of the upper estimate in (3.9), we use Theorem 2.1, (2.14), (3.15) and (1.4):

$$\begin{aligned}
\|r\mathcal{L}^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} |r(x)| \int_{-\infty}^{\infty} G(x, t) dt \\
&\leq 4 \sup_{x \in \mathbb{R}} \left(|r(x)| \int_{x-1}^{x+1} G(x, t) dt \right) = 4 \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} \int_{x-1}^{x+1} \frac{q(x)}{q(t)} \cdot q(t) G(x, t) dt \\
&\leq 4a \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} \int_{x-1}^{x+1} q(t) G(x, t) dt \leq 4a \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} \int_{-\infty}^{\infty} q(t) G(x, t) dt \\
&= 4a \sup_{x \in \mathbb{R}} \left[u(x) \int_{-\infty}^x v''(t) dt + v(x) \int_x^{\infty} u''(t) dt \right] \\
&= 4a \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} (v'(x)u(x) - u'(x)v(x)) = 4am_0(r, q).
\end{aligned}$$

To prove (3.11), we consecutively use Theorem 2.1, (2.14), (3.15), (1.4), (2.3), (2.4), (2.7) and (3.4):

$$\begin{aligned}
\left\| r \frac{d}{dx} \mathcal{L}^{-1} \right\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} |r(x)| \left| \frac{d}{dx} \int_{-\infty}^{\infty} G(x, t) dt \right| \\
&\leq 4 \sup_{x \in \mathbb{R}} |r(x)| \left[|u'(x)| \int_{x-1}^x v(t) dt + v'(x) \int_x^{x+1} u(t) dt \right] \\
&= 4 \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} \left[|u'(x)| \int_{x-1}^x v''(t) \frac{q(x)}{q(t)} q(t) dt + v'(x) \int_x^{x+1} u''(t) \frac{q(x)}{q(t)} q(t) dt \right] \\
&\leq 4a \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} \left[|u'(x)| \int_{-\infty}^x v''(t) dt + v'(x) \int_x^{\infty} u''(t) dt \right] \\
&\leq 8a \sup_{x \in \mathbb{R}} \left(\frac{|r(x)|}{q(x)} \cdot \frac{v'(x)}{v(x)} \cdot \frac{|u'(x)|}{u(x)} \cdot \rho(x) \right) = 8a \sup_{x \in \mathbb{R}} \left(\frac{|r(x)|}{q(x)} \cdot \frac{1 - \rho^2(x)}{4\rho(x)} \right) \\
&\leq 2a \sup_{x \in \mathbb{R}} \left(\frac{|r(x)|}{q(x)} \frac{1}{\rho(x)} \right) \leq 2\sqrt{2}a \sup_{x \in \mathbb{R}} \frac{|r(x)|}{q(x)} \left(\frac{1}{d_1(x)} + \frac{1}{d_2(x)} \right) \\
&\leq 8a^{3/2} \sup_{x \in \mathbb{R}} \frac{|r(x)|}{\sqrt{q(x)}} = 8a^{3/2} m_1(r, q).
\end{aligned}$$

The proof of (3.12) is obvious, and (3.13) follows from (3.12), (2.8) and the triangle inequality. \square

Consider the system of equations

$$\left. \begin{aligned} z_1(x) &= f_1(x) + (B_{11}z_1)(x) + (B_{22}z_2)(x) \\ z_2(x) &= f_2(x) + (B_{21}z_1)(x) + (B_{22}z_2)(x) \end{aligned} \right\}, \quad x \in \mathbb{R}, \quad (3.17)$$

where $f_k \in C(\mathbb{R})$, $k = 1, 2$, $B_{ij} : C(\mathbb{R}) \rightarrow C(\mathbb{R})$, $i, j = 1, 2$ are linear operators.

Lemma 3.7. *Suppose that we have the inequality*

$$\|B_{ij}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq \frac{1}{4}, \quad i, j = 1, 2. \quad (3.18)$$

Then the system (3.17) has a unique solution $\{z_1, z_2\}$ such that $z_k \in C(\mathbb{R})$, $k = 1, 2$, and

$$\|z_1\|_{C(\mathbb{R})} + \|z_2\|_{C(\mathbb{R})} \leq 2(\|f_1\|_{C(\mathbb{R})} + \|f_2\|_{C(\mathbb{R})}). \quad (3.19)$$

Proof. Let us write down (3.12) in vector form. Set

$$z(x) := \begin{cases} z_1(x) \\ z_2(x) \end{cases} ; \quad f(x) := \begin{cases} f_1(x) \\ f_2(x) \end{cases} ; \quad B := \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \quad (3.20)$$

Then the system (3.17) becomes

$$z(x) = f(x) + (Bz)(x), \quad x \in \mathbb{R}. \quad (3.21)$$

Denote by $C_2(\mathbb{R})$ the vector space of vector functions $z(x)$, $x \in \mathbb{R}$ with continuous components $z_k(x)$, $x \in \mathbb{R}$, $k = 1, 2$ (see (3.20)), equipped with the norm

$$\|z\|_{C_2(\mathbb{R})} = \|z_1\|_{C(\mathbb{R})} + \|z_2\|_{C(\mathbb{R})}. \quad (3.22)$$

Let us estimate the norm of the operator $B : C_2(\mathbb{R}) \rightarrow C_2(\mathbb{R})$:

$$\begin{aligned} \|Bz\|_{C_2(\mathbb{R})} &= \|B_{11}z_1 + B_{12}z_2\|_{C(\mathbb{R})} + \|B_{21}z_1 + B_{22}z_2\|_{C(\mathbb{R})} \\ &\leq (\|B_{11}z_1\|_{C(\mathbb{R})} + \|B_{12}z_2\|_{C(\mathbb{R})}) + (\|B_{21}z_1\|_{C(\mathbb{R})} + \|B_{22}z_2\|_{C(\mathbb{R})}) \\ &\leq \frac{1}{2}(\|z_1\|_{C(\mathbb{R})} + \|z_2\|_{C(\mathbb{R})}) = \frac{1}{2}\|z\|_{C_2(\mathbb{R})}. \end{aligned}$$

Therefore, the operator $B : C_2(\mathbb{R}) \rightarrow C_2(\mathbb{R})$ is a compressing operator, and the lemma is proved. \square

4. PROOF OF THE MAIN RESULT

Below we prove Theorem 1.1. Let us introduce an operator $A : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ by the formula

$$(Af)(x) \stackrel{\text{def}}{=} q(x) \int_{x-\varphi(x)}^x f(\xi)(\xi - x + \varphi(x))d\xi, \quad x \in \mathbb{R}, \quad f \in C(\mathbb{R}). \quad (4.1)$$

Lemma 4.1. *We have the inequalities:*

$$\|A\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq \frac{1}{36a}, \quad (4.2)$$

$$\|(E - A)^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq \frac{36}{35}. \quad (4.3)$$

Proof. Let $f \in C(\mathbb{R})$. Then we have the relations

$$\begin{aligned} \|Af\|_{C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} q(x) \left| \int_{x-\varphi(x)}^x f(\xi)(\xi - x + \varphi(x)) d\xi \right| \\ &\leq \sup_{x \in \mathbb{R}} \left[q(x) \int_{x-\varphi(x)}^x (\xi - x + \varphi(x)) d\xi \right] \|f\|_{C(\mathbb{R})} \leq \sup_{x \in \mathbb{R}} \left[\frac{(q(x)\varphi(x))^2}{q(x)} \right] \cdot \|f\|_{C(\mathbb{R})} \\ &\leq \sup_{x \in \mathbb{R}} (q(x)\varphi(x))^2 \cdot \|f\|_{C(\mathbb{R})} \leq \frac{1}{36a} \|f\|_{C(\mathbb{R})} \Rightarrow (4.2). \end{aligned}$$

Inequality (4.3) follows from (4.2) and the expansion of the operator $(E - A)^{-1}$ in powers of the operator A . \square

Let us introduce some more notation:

$$f(x), \quad z_1(x), \quad z_2(x) \quad - \quad \text{are functions from } C(\mathbb{R}), \quad (4.4)$$

$$g(x) = [(E - A)^{-1}f](x), \quad x \in \mathbb{R}, \quad (4.5)$$

$$(F_1 z_1)(x) = \sum_{n=1}^{\infty} (A^n(q z_1))(x), \quad x \in \mathbb{R}, \quad (4.6)$$

$$(F_2 z_2)(x) = [(E - A)^{-1}(q\varphi z_2)](x), \quad x \in \mathbb{R}. \quad (4.7)$$

Lemma 4.2. *We have the relations*

$$g(x) \in C(\mathbb{R}), \quad F_1 z_1 \in C(\mathbb{R}), \quad F_2 z_2 \in C(\mathbb{R}), \quad (4.8)$$

$$\|g\|_{C(\mathbb{R})} \leq \frac{36}{35} \|f\|_{C(\mathbb{R})}, \quad (4.9)$$

$$\|F_1 z_1\|_{C(\mathbb{R})} \leq \frac{\|z_1\|_{C(\mathbb{R})}}{70}, \quad \|F_2 z_2\|_{C(\mathbb{R})} \leq \frac{6}{35} \|z_2\|_{C(\mathbb{R})}. \quad (4.10)$$

Proof. Inclusions (4.8) follow from (4.1), (4.2), (4.3) and estimates (4.9) and (4.10). Estimate (4.9) follows from (4.3). Consider (4.10). We have

$$\begin{aligned} \|Aq z_1\|_{C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} q(x) \left| \int_{x-\varphi(x)}^x q(\xi)(\xi - x + \varphi(x)) z_1(\xi) d\xi \right| \\ &\leq \sup_{x \in \mathbb{R}} q(x)^2 \left[\int_{x-\varphi(x)}^x \frac{q(\xi)}{q(x)} (\xi - x + \varphi(x)) d\xi \right] \cdot \|z_1\|_{C(\mathbb{R})} \\ &\leq \frac{a}{2} \sup_{x \in \mathbb{R}} (q(x)\varphi(x))^2 \cdot \|z_1\|_{C(\mathbb{R})} \leq \frac{1}{72} \|z_1\|_{C(\mathbb{R})} \Rightarrow (\text{see (4.3)}) : \\ \|F_1 z_1\|_{C(\mathbb{R})} &\leq \sum_{n=1}^{\infty} \|A^n(q z_1)\|_{C(\mathbb{R})} \leq \sum_{n=1}^{\infty} \|A\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})}^{n-1} \cdot \|A(q z_1)\|_{C(\mathbb{R})} \\ &\leq \left(1 - \frac{1}{36a}\right)^{-1} \cdot \frac{\|z_1\|}{72} \leq \frac{36}{35} \frac{\|z_1\|_{C(\mathbb{R})}}{72} = \frac{\|z_1\|}{70} \Rightarrow (4.10). \end{aligned}$$

Similarly,

$$\begin{aligned} \|F_2 z_2\|_{C(\mathbb{R})} &= \sup_{x \in \mathbb{R}} \|(E - A)^{-1}(q\varphi z_2)\|_{C(\mathbb{R})} \\ &\leq \|(E - A)^{-1}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \cdot \frac{\|z_2\|_{C(\mathbb{R})}}{6\sqrt{a}} \leq \frac{6}{35} \|z_2\|_{C(\mathbb{R})} \quad \Rightarrow \quad (4.10). \end{aligned}$$

Consider the system (3.17), where we set (see (4.1), (2.10), (2.14), (4.4), (4.5), (4.6) and (4.7)):

$$f_1(x) = (Gg)(x), \quad f_2(x) = \frac{d}{dx}(Gg)(x), \quad x \in R, \quad (4.11)$$

$$(B_{11}z_1)(x) = -(GF_1z_1)(x), \quad x \in R, \quad (4.12)$$

$$(B_{12}z_2)(x) = -(GF_2z_2)(x), \quad x \in R, \quad (4.13)$$

$$(B_{21}z_1)(x) = -\left(\frac{d}{dx}GF_1z_1\right)(x), \quad x \in R, \quad (4.14)$$

$$(B_{22}z_2)(x) = -\left(\frac{d}{dx}GF_2z_2\right)(x), \quad x \in R. \quad (4.15)$$

□

Lemma 4.3. *We have the following estimates for the norms of the operator B_{ij} , $i, j = 1, 2$ (see (4.12), (4.13), (4.14) and (4.15)):*

$$\|B_{ij}\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \leq \frac{1}{4}, \quad i, j = 1, 2. \quad (4.16)$$

Proof. The assertion of the lemma follows from (2.14), (3.2), (3.3), (4.2), (4.3) and (4.10).

For example, for $i = j = 1$ and $i = j = 2$, respectively, we have

$$\begin{aligned} \|B_{11}z_1\|_{C(\mathbb{R})} &= \|GF_1z_1\|_{C(\mathbb{R})} \leq \|G\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \cdot \|F_1z_1\|_{C(\mathbb{R})} \leq \frac{1}{70} \|z_1\|_{C(\mathbb{R})} \quad \Rightarrow \quad (4.16); \\ \|B_{22}z_2\|_{C(\mathbb{R})} &= \left\| \frac{d}{dx}GF_2z_2 \right\|_{C(\mathbb{R})} \leq \left\| \frac{d}{dx}G \right\|_{C(\mathbb{R}) \rightarrow C(\mathbb{R})} \cdot \|F_2z_2\|_{C(\mathbb{R})} \\ &\leq \frac{6\sqrt{2}}{35} \|z_2\|_{C(\mathbb{R})} < \frac{1}{4} \|z_2\|_{C(\mathbb{R})} \quad \Rightarrow \quad (4.16). \end{aligned}$$

□

From Lemmas 4.3 and 3.7 it follows that the system of equations

$$\begin{cases} z_1(x) = [G(E - A)^{-1}f](x) - [GF_1z_1](x) + [GF_2z_2](x) \\ z_2(x) = \left[\frac{d}{dx}G(E - A)^{-1}f\right](x) - \left[\frac{d}{dx}GF_1z_1\right](x) + \left[\frac{d}{dx}GF_2z_2\right](x) \end{cases}, \quad x \in \mathbb{R} \quad (4.17)$$

has a unique solution $z(x) \in C_2(\mathbb{R})$ (see (3.20)), and (see (4.6), (3.2), (3.3), (3.19), (4.4) and (4.11)), we have

$$\|z_1\|_{C(\mathbb{R})} + \|z_2\|_{C(\mathbb{R})} \leq \left[\|G(E - A)^{-1}f\|_{C(\mathbb{R})} + \left\| \frac{d}{dx} G(E - A)^{-1}f \right\|_{C(\mathbb{R})} \right] \leq c\|f\|_{C(\mathbb{R})}. \quad (4.18)$$

Note that there is a relationship between the functions $z_1(x)$ and $z_2(x)$ that can be checked in a straightforward way (see (4.17)):

$$z_2(x) = z_1'(x), \quad x \in \mathbb{R}.$$

Denote

$$y(x) = z_1(x), \quad x \in \mathbb{R} \quad \Rightarrow \quad y'(x) = z_1'(x) = z_2(x), \quad x \in \mathbb{R}. \quad (4.19)$$

By (4.19), the first equation in (4.17) and the estimate (4.18) take the form (4.20) and (4.21), respectively:

$$y(x) = (G(E - A)^{-1}f)(x) - (GF_1y)(x) + (GF_2y)(x), \quad x \in \mathbb{R}, \quad (4.20)$$

$$\|y\|_{C(\mathbb{R})} + \|y'\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}. \quad (4.21)$$

From (4.20) it follows that

$$y(x) = (Gw)(x), \quad x \in \mathbb{R}; \quad w(x) = ((E - A)^{-1}f)(x) - (F_1y')(x) + (F_2y)(x), \quad (4.22)$$

and (see (4.9), (3.2), (3.3), (4.21), we have

$$\|w\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}. \quad (4.23)$$

Thus, by Theorems 2.6, 2.7 and 3.5, (4.22) and (4.23), we get

$$-y''(x) + q(x)y(x) = w(x) = ((E - A)^{-1}f)(x) - (F_1y)(x) + (F_2y')(x), \quad x \in \mathbb{R}, \quad (4.24)$$

$$\|y''\|_{C(\mathbb{R})} + \|qy\|_{C(\mathbb{R})} \leq c\|w\|_{C(\mathbb{R})} \leq c\|f\|_{C(\mathbb{R})}. \quad (4.25)$$

Since $qy \in C(\mathbb{R})$ (see (4.21) and (4.25)) and

$$((E - A)^{-1}qy)(x) = q(x)y(x) + (F_1y)(x), \quad x \in \mathbb{R},$$

we obtain, by combining the last equality with (4.24), that

$$-y''(x) = (E - A)^{-1}[f(x) - q(x)y(x) + q(x)\varphi(x)y'(x)], \quad x \in \mathbb{R}.$$

But $y'' \in C(\mathbb{R})$, and therefore

$$\begin{aligned} -y''(x) + (Ay'')(x) &= f(x) - q(x)y(x) + q(x)\varphi(x)y'(x), \quad x \in \mathbb{R} \quad \Rightarrow \\ -y''(x) &= q(x) \left[y(x) - \varphi(x)y'(x) + \int_{x-\varphi(x)}^x y''(\xi)(\xi - x + \varphi(x))d\xi \right] = f(x), \quad x \in \mathbb{R}. \end{aligned} \quad (4.26)$$

From (4.26) and (2.16) we obtain (1.1), i.e., $y(x)$, $x \in \mathbb{R}$, is a solution of (1.1), and we have the estimate (1.3) (see (4.21)). The uniqueness of such a solution (1.1) follows from the linearity of this equation and the estimate (1.3). From (4.25) and the triangle inequality, we get (1.5). \square

Proof of Corollary 1.2. The estimate (1.7) follows from Theorem 1.1 and (4.25). \square

5. EXAMPLE

Below we consider equation (1.1) with

$$q(x) = 2(1 + x^2) + (1 + x^2) \sin(|x|^2), \quad x \in \mathbb{R}. \quad (5.1)$$

The function (5.1) satisfies (1.2), and with the help of Theorem 1.1, we show that such an equation (1.1) is correctly solvable in the space $C(\mathbb{R})$ if $\sigma := \frac{1}{31}$ (see (1.5)). To prove this fact, we use the following simple lemma, which can be useful for checking condition (1.4).

Lemma 5.1. *Suppose that we are given a function $q(x)$, $x \in \mathbb{R}$, and (1.2) holds. If there is a positive, continuously differentiable function $q_1(x)$ for $x \in \mathbb{R}$ such that*

1) *for all $x \in \mathbb{R}$, we have the inequalities*

$$\nu^{-1}q_1(x) \leq q(x) \leq \nu q_1(x) \quad (5.2)$$

where the constant $\nu \in [1, \infty)$ does not depend on the choice of a point $x \in \mathbb{R}$;

2) *$s < \infty$ where*

$$s = \sup_{x \in \mathbb{R}} \frac{|q_1'(x)|}{q_1(x)}. \quad (5.3)$$

Then the function q satisfies condition (1.4) for $a = \nu^2 e^s$.

Proof. Let $t \in [x - 1, x + 1]$, $x \in \mathbb{R}$. In the following relations, we use (5.2) and (5.3):

$$\begin{aligned} \left| \ln \frac{q_1(t)}{q_1(x)} \right| &= |\ln q_1(t) - \ln q_1(x)| = \left| \int_x^t \frac{q_1'(\xi)}{q_1(\xi)} d\xi \right| \\ &\leq \left| \int_x^t \frac{|q_1'(\xi)|}{q_1(\xi)} d\xi \right| \leq s|x - t| \leq s \quad \Rightarrow \\ e^{-s} &\leq \frac{q_1(t)}{q_1(x)} \leq e^s, \quad |t - x| \leq 1 \quad \Rightarrow \\ \frac{q(t)}{q(x)} &= \frac{q(t)}{q_1(t)} \cdot \frac{q_1(t)}{q_1(x)} \cdot \frac{q_1(x)}{q(x)} \leq \nu^2 e^s, \quad |t - x| \leq 1, \\ \frac{q(t)}{q(x)} &= \frac{q(t)}{q_1(t)} \cdot \frac{q_1(t)}{q_1(x)} \cdot \frac{q_1(x)}{q(x)} \geq \frac{1}{\nu^2} e^{-s}, \quad |t - x| \leq 1. \end{aligned}$$

In the case (5.1), clearly, $q_1(x) = 1 + x^2$, $x \in \mathbb{R}$, because

$$\frac{1 + x^2}{3} < 1 + x^2 \leq 2(1 + x^2) + (1 + x^2) \sin(|x|^2) \leq 3(1 + x^2), \quad x \in \mathbb{R},$$

and therefore $\nu = 3$. In addition, $s = 1$ because

$$s = \sup_{x \in \mathbb{R}} \frac{|q_1'(x)|}{q_1(x)} = \sup_{x \in \mathbb{R}} \frac{2|x|}{1 + x^2} \leq 1.$$

Hence

$$\frac{1}{6\sqrt{a}} = \frac{1}{6} \cdot \frac{1}{3\sqrt{e}} = \frac{1}{18\sqrt{e}} \geq \frac{1}{31} = \sigma,$$

as required. □

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